Vibrations in a Cold Electroactive Dipole Lattice

RICHARD L. LIBOFF
Courant Institute of Mathematical Sciences and Physics Department, New York University, New York, New York
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The propagation of small amplitude disturbances in a permanent dipole lattice about an equilibrium configuration which includes a steady electric and dipole field, parallel to one another, is considered. The governing dispersion relation is sixth order in ω^2. Two of the six modes are degenerate. The remaining four split into two isotropic modes and two anisotropic modes. Specific forms are uncovered for these roots together with explicit relations for the components of the related twelve-dimensional state vector.

I. INTRODUCTION AND SUMMARY OF RESULTS

In this paper we initiate a study of the disturbances which may propagate in an incompressible, cold, electroactive, dipole lattice. The term electroactive indicates that the equilibrium configuration includes a steady electric field. Most previous investigations of the propagation of electromagnetic radiation through dielectric media are either phenomenologically oriented or stem from a limited kinetic approach.[1–3] The constraint of coldness, on the other hand, which implies that there be no spread in dipole orientation space in addition to the standard demand that there be no spread in velocity space, permits one to write down simplified nonphenomenological equations of motion. These equations are very similar to those used to investigate the modes of excitation of a magnetic dipole lattice in the limit of vanishing photon coupling and exchange energy[4,5] (with the roles of magnetic and electric dipoles and fields interchanged).

More directly the included theory is equally well suited for a cold incompressible uniform gas of permanent dipoles. However, inasmuch as the constraint of uniformity of sites is more easily realized in a solid than in a gas, the present analysis is oriented toward the theory of solids.

The state vector for the included system is twelve dimensional. Its components are the four vectors, dipole rotation velocity $\mathbf{\Omega}$, dipole density $\mathbf{\Pi}$, electric field $\mathbf{E}$, and magnetic field $\mathbf{B}$. Owing to the cold-incompressible constraint, Newton's second law is decoupled from the complement of equations of motion in a manner such that the time development of the displacement vector is determined, granted that $\mathbf{E}$ and $\mathbf{\Pi}$ are known. This equation is omitted in favor of the torque equation, which in the absence of any stress mechanism relates the time development of $\mathbf{\Pi}$ to $\mathbf{\Pi}$ and $\mathbf{E}$ only.

The aggregate of equations so obtained is linearized about an equilibrium configuration which includes steady electric ($\mathbf{E}_0$) and dipole-density ($\mathbf{\Pi}_0$) fields, and zero rotation and magnetic vectors. A plane-wave analysis yields two classes of waves. The first class is isotropic while the second class is anisotropic and depends on the angle between the propagation vector $\mathbf{k}$ and the steady electric field $\mathbf{E}_0$. All waves are such that for any given frequency there are two characteristic speeds of propagation.

More precisely, the emergent dispersion relation is sixth order in ω^2, this owing to (a) the time reversability (in present context: no friction) of the starting equations, and (b) each of the twelve scalar components of the state vector satisfies a first-order equation in time. Two of the six modes are degenerate (viz., $\omega^2=0$ twice). For the first root $\Omega_{\parallel}$ is constant with all other components of the state vector vanishing, and for the second degenerate root $(E_{\parallel}/E_{\perp})=(k_{\parallel}/k_{\perp})$ with all other components vanishing. [The notation is such that components with subscript $\parallel$ are in the direction of $\mathbf{E}_0$, components with the subscript $\perp$ and superscript k are perpendicular to E_0 and lie in the (E_0,k) plane; cf. Fig. 1.]

The isotropic mode is quadratic in ω^2 and relates to a state vector with nonvanishing $E_{\perp}$ and $\Omega_{\perp}^k$. For both frequencies large and small compared to Ω_0 most of the energy of this mode is concentrated in the electric field (as opposed to kinetic rotational energy). The natural frequency $\Omega_0=(\chi\epsilon_0E_0^2/nI)^{1/2}$, where χ is electric susceptibility, n is number density, and I is moment of inertia of an elementary dipole. For frequencies close to Ω_0 the energy of the mode is purely rotational. The index of refraction sketched as a function of Ω (cf. Fig. 2) readily indicates that only frequencies less than Ω_0 propagate undamped. The effect is analogous to the Faraday effect[6] in magnetoactive conductive media. There the radiation field is absorbed in Larmor resonance as opposed to the present instance where the radiation field is absorbed in electric-dipole resonance. The form of this isotropic dispersion relation is identical to the one uncovered by Born and Huang[7] in a semi-

[1] P. Debye, *Polar Molecules* (Dover Publications, Inc., New York, 1929), cf. Chap. 5.
[2] C. P. Smyth, *Dielectric Behavior and Structure* (McGraw-Hill Book Company, Inc., New York, 1955).
[3] A. R. von Hippel, *Dielectrics and Waves* (John Wiley and Sons, Inc., New York, 1954).
[4] C. Kittel, *Quantum Theory of Solids* (John Wiley & Sons, Inc., New York, to be published).
[5] L. Walker, Phys. Rev. **105**, 390 (1957).
[6] A. Sommerfeld, *Optics* (Academic Press Inc., New York, 1954).
[7] M. Born and K. Huang, *Dynamic Theory of Crystal Lattices* (Oxford University Press, London, 1956), cf. Eq. (8–23), p. 94.

phenomenological theory of the propagation of waves in a dipole lattice.

The anisotropic mode is also quadratic in ω^2 and relates to nonvanishing $[E_\perp^k, E_\parallel, \Omega_\perp]$. Writing the index of refraction as a function of ω (cf. Fig. 4) again indicates the presence of a cutoff frequency. In the direction $\theta=0$ the anisotropic index of refraction reduces to the isotropic index of refraction, with $E_\parallel=0$ and $(E_\perp^k, \Omega_\perp)$ related in the same manner as $(E_\perp, \Omega_\perp^k)$ are related for the isotropic case, and with cutoff frequency increasing until finally at $\theta=\pi/2$ the plane polarized wave $\mathbf{E}=(E_\parallel,0,0)$ propagates transversely with the speed of light in vacuum. In this limit there is no torque on the equilibrium dipoles so that they remain unperturbed.

The wave speed written as a function of wave number k is double-valued, which for fixed k, gives rise to "fast" and "slow" waves. The wave normal surfaces of these dispersive waves are surfaces of revolution about the $\mathbf{E}_0$ axis (cf. Fig. 5). In the extreme of small wavelength the fast wave becomes an isotropic vacuum electrodynamic wave while the slow wave becomes a nonpropagating oscillation with frequency $\sim\Omega_0$. For long wavelengths the slow wave propagates with the speed of light normal to the steady E_0 field while the fast wave becomes an isotropic, purely oscillatory fluctuation with frequency $\Omega_0(\kappa)^{1/2}$, where $\kappa=1+\chi$, is the specific inductive capacity.

II. ANALYSIS

1. Equations of Motion

The equations of motion which govern a cold incompressible (permanent) dipole gas appear as:

$$nI d\mathbf{\Omega}/dt=\mathbf{\Pi}\times\mathbf{E}, \quad (1)$$

$$d\mathbf{\Pi}/dt=\mathbf{\Omega}\times\mathbf{\Pi}, \quad (2)$$

$$\nabla\times\mathbf{B}=\mu_0 d\mathbf{\Pi}/dt+(d\mathbf{E}/dt)/c^2, \quad (3)$$

$$\nabla\times\mathbf{E}=-d\mathbf{B}/dt. \quad (4)$$

Equation (1) is the torque equation. The rotation vector $\mathbf{\Omega}$ is the angular velocity of an infinitesimal dipole element. The dipole strength per unit volume associated with the same element of media is $\mathbf{\Pi}$, while $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic field strengths, respectively. Dipole number density is n (it is constant) and I is the moment of inertia of an elemental dipole. Both kinetic and interparticle stress have been neglected in Eq. (1). Equation (2) is kinematical and ascribes the time rate of change $\mathbf{\Pi}$ to rotation only. This is consistent with incompressibility and the fact that the elemental dipoles are permanent in the present model.[8] However, the direction which the vector $\mathbf{\Pi}$ assumes is, of course, grossly influenced by $\mathbf{E}$.

Equations (4) and (5) are Maxwell's equations excluding the presence of "true" charge.

Newton's second law is conspicuous by its absence; however, it merely serves to introduce another fluid variable, i.e., the displacement which in turn is determined in time if $\mathbf{\Pi}$ and $\mathbf{E}$ are known. The system (1)–(4) is a closed set and we will satisfy ourselves with the examination of this reduced set of equations.

Specifically let us consider the perturbation solution to these equations about the equilibrium

$$\mathbf{E}_{\text{eq}}=\mathbf{E}_0;\quad \mathbf{\Pi}_{\text{eq}}=\mathbf{\Pi}_0,\quad \mathbf{\Omega}_{\text{eq}}=0,\quad \mathbf{B}_{\text{eq}}=0, \quad (5)$$
$$\mathbf{\Pi}_0=\epsilon_0\chi\mathbf{E}_0.$$

The specific inductive capacity is χ.

The perturbation variables $(\mathbf{E},\mathbf{B},\mathbf{\Omega},\mathbf{\Pi})$ satisfy the following equations,

$$nI d\mathbf{\Omega}/dt=\epsilon_0\chi\mathbf{E}_0\times\mathbf{E}+\mathbf{\Pi}\times\mathbf{E}_0, \quad (6)$$

$$d\mathbf{\Pi}/dt=\epsilon_0\chi\mathbf{\Omega}\times\mathbf{E}_0, \quad (7)$$

$$c^2\nabla(\nabla\cdot\mathbf{E})-c^2\nabla^2\mathbf{E}+d^2\mathbf{E}/dt^2+(d^2\mathbf{\Pi}/dt^2)/\epsilon_0=0. \quad (8)$$

2. Dispersion Relations

The plane wave transform (viz., $\exp\{i\mathbf{k}\cdot\mathbf{x}-i\omega t\}$) of the latter three equations appears as

$$i\omega\epsilon_0\chi\mathbf{E}_0\times\mathbf{E}+\epsilon_0\chi\mathbf{E}_0\times(\mathbf{\Omega}\times\mathbf{E}_0)-\omega^2 nI\mathbf{\Omega}=0, \quad (9)$$

$$(c^2k^2-c^2\mathbf{kk}-\omega^2)\mathbf{E}+i\omega\chi\mathbf{E}_0\times\mathbf{\Omega}=0. \quad (10)$$

The dyad $\mathbf{kk}$ operating on $\mathbf{E}$ gives $\mathbf{k}(\mathbf{k}\cdot\mathbf{E})$. In the representation in which $\mathbf{E}_0=(E_0,0,0)$ and $k=(k_\parallel,k_\perp,0)$, the secular equation which Eqs. (9) and (10) gives appears as

$$\begin{array}{cccccc} \Omega_\parallel & \Omega_\perp^k & E_\perp & E_\parallel & E_\perp & \Omega_\perp \end{array}$$
$$\left|\begin{array}{cccccc} -\omega^2/\Omega_0^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & [1-(\omega^2/\Omega_0^2)] & [i\omega/E_0] & 0 & 0 & 0 \\ 0 & -i\omega\chi E_0 & c^2k^2-\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (-i\omega/E_0) & [1-(\omega^2/\Omega_0^2)] \\ 0 & 0 & 0 & (c^2k_\perp^2-\omega^2) & -c^2k_\perp k_\parallel & 0 \\ 0 & 0 & 0 & -c^2k_\parallel k_\perp & [c^2k_\parallel^2-\omega^2] & i\omega\chi E_0 \end{array}\right|=0. \quad (11)$$

[8] If the dipoles were polarizable Eq. (2) would include a forcing term $\sim\mathbf{E}$.

The frequency Ω_0 has been written for

$$\Omega_0^2=\epsilon_0\chi E_0^2/nI. \tag{12}$$

The notation[9] is such that a $\perp$ subscript denotes a component perpendicular to both $\mathbf{E}_0$ and $\mathbf{k}$. The $\perp$ subscript combined with the $\mathbf{k}$ superscript appears on components perpendicular to $\mathbf{E}_0$ and in this plane of $\mathbf{k}$ and $\mathbf{E}_0$. The vector $\mathbf{\Omega}$ is sketched in this representation in Fig. 1.

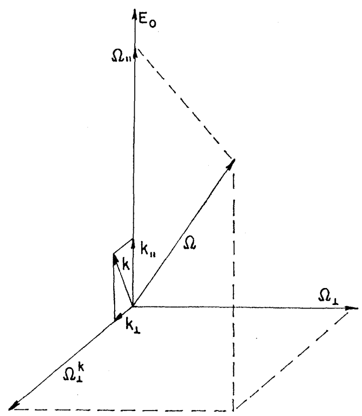

FIG. 1. The vector $\mathbf{\Omega}$ in the $\mathbf{E}_0,\mathbf{k}$ representation.

Equation (11) indicates that the dispersion formula is twelfth order in ω which relates to the fact that each of the four vectors $(\mathbf{\Omega},\mathbf{\Pi},\mathbf{E},\mathbf{B})$ satisfy a first-order equation in time. Equation (11) when combined with Eqs. (8) and (4) determine the complete nature of the state vector $\mathbf{\Gamma}=(\mathbf{E},\mathbf{B},\mathbf{\Omega},\mathbf{\Pi})$ for any of the six ω^2 roots.

The first pair of roots $\omega^2=0$ relates to a steady value of $\Omega_{\parallel}$ with all other components of $\mathbf{\Gamma}$ zero.

The remaining roots separate into two classes. In the isotropic class,

$$c^2k^2/\omega^2=n_{\text{iso}}^2=1+[\chi\Omega_0^2/(\Omega_0^2-\omega^2)], \tag{13}$$

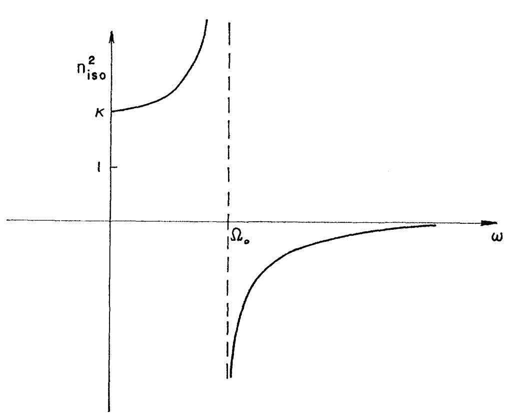

FIG. 2. The isotropic index of refraction as a function of ω.

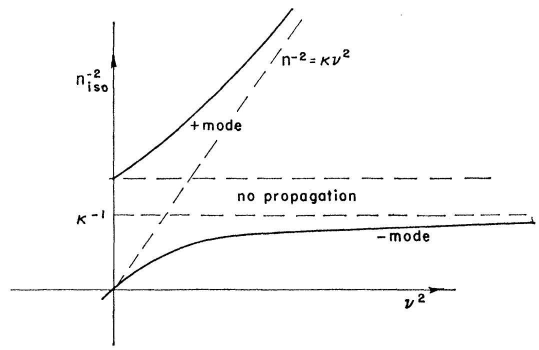

FIG. 3. The isotropic index of refraction as a function of $\nu^2=\Omega_0^2/c^2k^2$.

the inverse of which is given by

$$2[\omega_\pm^2/c^2k^2]_{\text{iso}}=(1+\kappa v^2)\left\{1\pm\left[1-\frac{4v^2}{(1+\kappa v^2)^2}\right]^{1/2}\right\}. \tag{14}$$

The specific inductive capacity $\kappa=1+\chi$, and the nondimensional velocity $v^2=\Omega_0^2/c^2k^2$. For this class of roots $\Omega_\perp^k$ and $E_\perp$ are nonzero. Their ratio in this mode of ex-

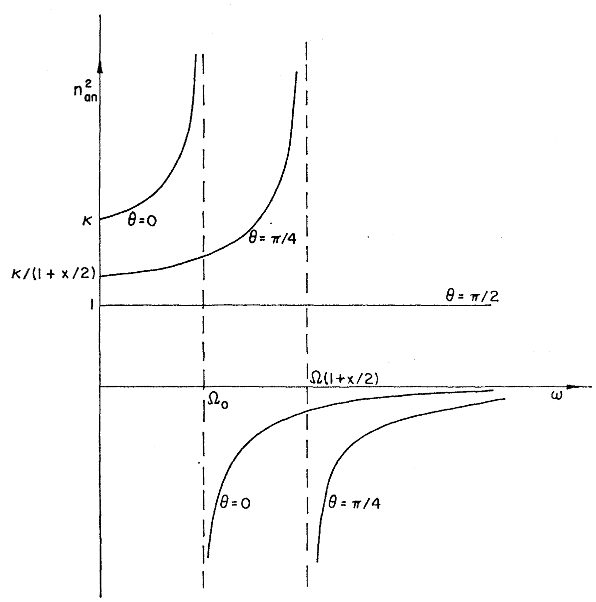

FIG. 4. The anisotropic index of refraction for three different values of θ.

citation is given by

$$\left[\frac{\chi\epsilon_0E^2}{nI\Omega^2}\right]^{1/2}=\frac{E_\perp/E_0}{\Omega_\perp/\Omega_0}=i\left[\frac{1-(\omega^2/\Omega_0^2)}{\omega/\Omega_0}\right]. \tag{15}$$

The electric and rotational fields are $(\pi/2)$ radians out of phase for $\omega<\Omega_0$, and $-\pi/2$ radians out of phase for $\omega>\Omega_0$. For both $\omega\gg\Omega_0$ and $\omega\ll\Omega_0$ most of the energy of the mode is concentrated in the electric field (as opposed to kinetic rotational energy).

[9] $k_\perp$ has been written for $k_\perp^k$.

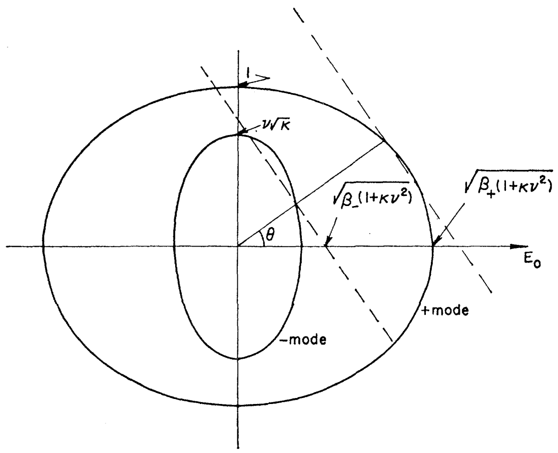

FIG. 5. Plane-wave speed versus θ; $+$=fast wave, $-$=slow wave. Dashed lines represent orientation of wave fronts at time c^{-1}, at the inclination θ.

The index of refraction as given by Eq. (13) is sketched in Fig. 2.

It is quite clear that in this mode only frequencies less than Ω_0 propagate. In Fig. 3, n^{-2} is sketched as a function of $v^2=\Omega_0^2/c^2k^2$ with part of the image of the forbidden region of Fig. 2 shown (viz., "no propagation"). Some of the velocities which do not appear as solutions in Fig. 2 appear in Fig. 3 as the asymptotic segment of the positive branch. These are nonpropagating purely oscillatory roots: $\omega^2=\kappa\Omega_0^2$. These modes appear in the limit of vanishing wave number (large wavelengths). The concentration of field energy to rotational energy in this limit goes as $|(1-\kappa)/(\kappa)^{1/2}|$ whence most of the energy is in kinetic rotational form which is as expected.

The remaining factor of the dispersion equation (11) governing the substate vector $[E_{\shortparallel},E_\perp,\Omega_\perp]$ is a cubic in ω^2, which in turn contains a ω^2 factor. The eigenvector which relates to this degenerate $\omega^2=0$ mode is given by

$$\Omega_\perp=0;\quad [E_{\shortparallel}/E_\perp]=[k_{\shortparallel}/k_\perp], \tag{16}$$

together with $\Omega_\perp{}^k$, $\Omega_{\shortparallel}$, $E_\perp{}^k$ all equal to zero. These latter three components also vanish for the remaining anisotropic modes which give rise to the index of refraction,

$$n_{\text{an}}{}^2=1+\frac{\chi\Omega_0{}^2\cos^2\theta}{\Omega_0{}^2(1+\chi\sin^2\theta)-\omega^2}. \tag{17}$$

The inverse (i.e., ω as a function of k) appears as,

$$2W_\pm{}^2\equiv 2(\omega_\pm{}^2/c^2k^2)_{\text{an}}$$
$$=(1+\kappa v^2)\left\{1\pm\left[1-\frac{4v^2(1+\chi\sin^2\theta)}{(1+\kappa v^2)^2}\right]^{1/2}\right\}. \tag{18}$$

The eigenvector which corresponds to these modes of excitation is given by

$$\frac{E_\perp{}^k/E_0}{\Omega_\perp/\Omega_0}=-i\left[\frac{1-(\omega^2/\Omega_0{}^2)}{\omega/\Omega_0}\right], \tag{19}$$

$$[E_{\shortparallel}/E_\perp{}^k]=c^2k_\perp k_{\shortparallel}/(c^2k_\perp{}^2-\omega^2). \tag{20}$$

This class of waves is purely transverse in the rotation field Ω. The anisotropic index of refraction as given by Eq. (17) is sketched for three different values of θ in Fig. 4.

For propagation normal to the steady E_0 field the equations become degnerate. One of the roots $\omega^2=\kappa\Omega_0^2$ is nonpropagating and corresponds to the eigenvector (19), while the remaining root $\omega^2=c^2k^2$ corresponds to $E_{\shortparallel}\neq 0$, $E_\perp{}^k=\Omega_\perp=0$. This latter root is the vacuum electrodynamic mode; it is plane polarized, and purely transverse (recall that $E_\perp=0$ also).

3. Wave-Normal Surfaces

It is instructive at this point to include a brief discussion of the surfaces formed by plotting the plane-wave speed (ω/k) against θ in polar coordinates. The complete three-dimensional locus of such speeds (wave-normal surface) is then obtained by rotating this curve about the $\mathbf{E}_0$ axis. A sketch of the two anisotropic sur-

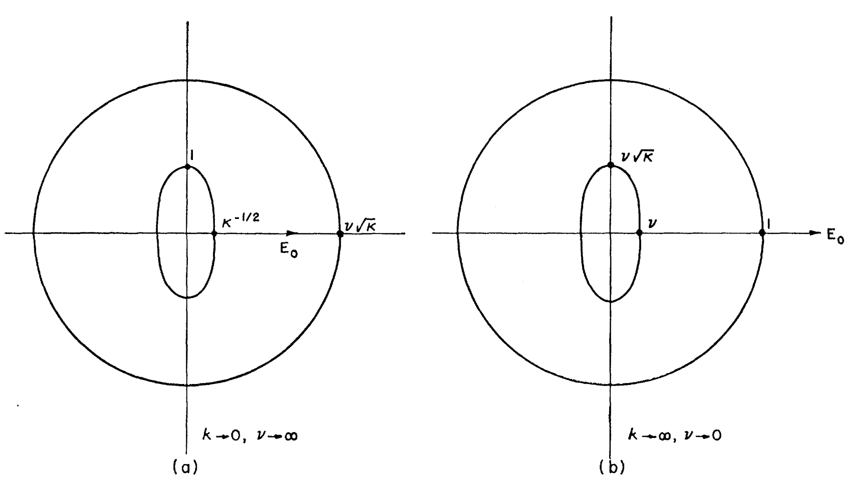

FIG. 6. Plane-wave speed versus θ in the two extremes $k\rightarrow 0$ and $k\rightarrow\infty$.

faces ("fast" and "slow" waves) so formed is shown in Fig. 5.

In these diagrams the functions $\beta_\pm$ are given by

$$2\beta_\pm = 1 \pm \left[1 - \left(\frac{2v}{1+\kappa v^2}\right)^2\right]^{1/2}.$$

In both extremes of vanishing and very large wave number the fast wave becomes isotropic. For small wavelengths the fast wave becomes the vacuum electrodynamic mode $\omega^2 = c^2k^2$, while for large wavelengths the fast wave collapses to the nonpropagating mode $\omega^2 = \kappa\Omega_0^2$. Similarly the slow wave, in the limit of small wavelengths, becomes a nonpropagating anisotropic wave $\omega^2 = \Omega_0^2(1+\chi \sin^2\theta)$, while in the limit of large wavelengths it becomes a propagating anisotropic wave $(\omega^2/k^2) = (c^2/\kappa)(1+\chi \sin^2\theta)$. These surfaces are sketched in Fig. 6.

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Quantum Theory of Domain-Wall Motion*

JAMES F. JANAK
Laboratory for Insulation Research, Massachusetts Institute of Technology, Cambridge, Massachusetts
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Several workers have examined the enhancement of nuclear magnetic resonance within a Bloch wall, and have demonstrated the existence of both bound and free "spin-wave" excitations on the Bloch wall structure. The free states correspond to precessional excitations akin to ordinary spin-wave excitations, while the bound states form a convenient basis for the representation of domain-wall motion. We derive the spectrum of both types of excitations, including exchange, anisotropy, and dipole field contributions for an infinite uniaxial ferromagnet. In contrast to earlier treatments, we treat the dipole field exactly (in the magnetostatic approximation), and show that this leads to a translational spectrum in which many states are degenerate with the "uniform translation," which is the translational mode excited by a uniform external magnetic field. The existence of such degeneracy is required for damping by imperfections to occur. The precessional spectrum is greatly different from the usual spin-wave spectrum, and, in particular, is not a symmetric function of **k**. The dipole fields lead to strong interactions, not conserving momentum, between the precessional modes; such interactions may explain the increase in ferromagnetic-resonance linewidth which is observed experimentally in the presence of a domain wall (in low dc magnetic fields). The motion of the domain wall, when it is bound to a certain position in the crystal by linear restoring forces, is studied by a Green's function technique. The domain-wall effective mass so obtained is identical to the expression given by Döring, and the domain-wall damping parameter proves to be simply related to the energy dispersion of the uniform translational mode. We calculate this energy dispersion due to scattering by the dipole fields, and due to "fluctuations," as used by Clogston *et al.* to explain the linewidth in disordered systems, such as the ferrites. The damping due to intrinsic scattering processes is proportional to T^2, while the damping due to "fluctuations" is essentially temperature-independent. In disordered systems, such as ferrite, the resonance linewidth and domain-wall damping due to "fluctuations" should agree to within a factor of order unity. The motion is *not* describable by the Landau-Lifshitz equation. This communication is intended to demonstrate that a formulation for the quantum-mechanical study of domain-wall motion exists, and has the properties necessary to explain the losses which occur during such motion; it is not intended to lead to any quantitative results which can be directly compared with experiment. We also consider the specific heat contribution due to the domain wall, and we find that this is proportional to T above about 10^{-2} °K. It should be possible to observe such a specific heat contribution in YIG below 1°K.

I. INTRODUCTION

SEVERAL workers[1–4] have considered the "spin-wave" excitations on the Bloch wall structure, both in ferro- and antiferromagnetic systems. It appears to be generally true that there exist two types of these excitations: Those bound to the wall, corresponding to translation of the wall (these all tend to zero well into the domains); and those which tend to plane waves well into the domains, corresponding to precessional modes in the domain-wall (DW) configuration. Previous work with these excitations has been aimed at evaluating the contribution to the nuclear magnetic resonance linewidth due to the presence of the Bloch wall; the

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